THE SEMI-LINEAR TORSIONAL RIGIDITY ON A COMPLETE RIEMANNIAN TWO-MANIFOLD

JIE XIAO

ABSTRACT. This note is concerned with some essential properties (optimal isoperimetry, first variation, and monotonicity formula) of the so-called $[0,1) \ni \gamma$-torsional rigidity $\mathcal{T}_{\gamma,\mathrm{g}}$ on a complete Riemannian two-manifold $(\mathbb{M}^2, \mathrm{g})$. Even in the special case of $\mathbb{R}^2$, major results are new.

1. INTRODUCTION

Throughout this note, on $(\mathbb{M}^2, \mathrm{g})$ – a two-dimensional manifold $\mathbb{M}^2$ with a complete Riemannian metric g, we denote by

$$d_{\mathrm{g}}(\cdot,\cdot);\ \ \langle\cdot,\cdot\rangle_{\mathrm{g}};\ \ |\cdot|_{\mathrm{g}};\ \ K_{\mathrm{g}}(\cdot,\cdot);\ \ dA_{\mathrm{g}}(\cdot);\ \ dL_{\mathrm{g}}(\cdot);\ \ \Delta_{\mathrm{g}}(\cdot);\ \ \nabla_{\mathrm{g}}(\cdot),$$

the distance function; the inner product between two vectors in the tangent bundle; the norm of a vector; the Gauss curvature; the area element; the length element; the Laplace-Beltrami operator; the gradient, respectively. Moreover, $B_{\mathrm{g}}(o,r) = \{z \in \mathbb{M}^2 : d_{\mathrm{g}}(z,o) < r\}$ denotes the geodesic disk centered at o with radius r, and the isoperimetric constant of $(\mathbb{M}^2, \mathrm{g})$ is determined by

$$\tau_{\mathrm{g}} = \inf_{O\in\mathcal{F}(\mathbb{M}^2)} \frac{\big(L_{\mathrm{g}}(\partial O)\big)^2}{A_{\mathrm{g}}(O)}.$$

When $\mathbb{M}^2$ is the flat Euclidean plane $\mathbb{R}^2$, we naturally equip it with the standard Euclidean metric e and therefore the previous notations will be changed correspondingly, i.e., g is replaced by e. In particular, $\tau_{\mathrm{e}} = 4\pi$.

For a parameter $\gamma \in [0,1)$ and a relatively compact domain $O \subseteq \mathbb{M}^2$ with C^∞ smooth boundary ∂O, denoted by $O \in \mathcal{F}(\mathbb{M}^2)$, let u be the solution of the following semi-linear boundary value problem (see [18], [6], [8], [7], [4], and their related references for the Euclidean case $\mathbb{R}^2$):

$$\begin{cases} \Delta_{\mathrm{g}} u = -u^\gamma \ \ \&\ \ u > 0 & \text{in} \quad O; \\ \qquad\qquad\qquad u = 0 & \text{on} \quad \partial O, \end{cases} \tag{1}$$

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where the second identity follows from Green's theorem. Then the semi-linear (or γ-) torsional rigidity of O as the cross section of the cylindrical beam $O\times\mathbb{R}$ is defined as

$$\mathcal{T}_{\gamma,\mathrm{g}}(O)=\int_O\left|\nabla_{\mathrm{g}}u\right|^2_{\mathrm{g}}\,dA_{\mathrm{g}}=\int_O u^{1+\gamma}\,dA_{\mathrm{g}}.$$

Note that if $\gamma=0$ then (1) is just the classical torsion problem and the resulting 0-torsional rigidity is standard. As well-known, under $\gamma=1$ the problem (1) has more than one non-trivial solutions, and thus the following eigenvalue problem is instead considered:

$$\begin{cases}\Delta_{\mathrm{g}}u=-\lambda u\ \ \&\ \ u>0 & \text{in}\quad O;\\ \qquad\qquad\qquad\quad u=0 & \text{on}\quad \partial O,\end{cases}\tag{2}$$

whose principal (or first) eigenvalue is determined through

$$\Lambda_{\mathrm{g}}(O):=\inf_{v\in W_0^{1,2}(O)}\left\{\int_O\left|\nabla_{\mathrm{g}}v\right|^2_{\mathrm{g}}\,dA_{\mathrm{g}}:\quad\int_O v^2\,dA_{\mathrm{g}}=1\right\},$$

where $W_0^{1,2}(O)$ stands for the Sobolev space of all compactly-supported C^∞ functions v on O with v^2 and $|\nabla_{\mathrm{g}}v|^2_{\mathrm{g}}$ being dA_{g}-integrable on O.

On the basis of Section 5 – a γ-torsional rigidity Schwarz's lemma for the conformal mappings on $\mathbb{R}^2$, we shall present some fundamental properties of $\mathcal{T}_{\gamma,\mathrm{g}}$ in: Section 2 – the optimal isoperimetric inequality in terms of τ_{g}; Section 3 – the first variational formula arising from a domain deformation; Section 4 – the monotonicity for the γ-torsional rigidity of a geodesic disk.

2. ISOPERIMETRY

Whenever $\mathbb{M}^2=\mathbb{R}^2$, a famous problem posed by St. Venant in 1956 and settled by G. Pólya in 1948 (cf. [16, p. 121]) was to prove that among all simply connected domains of given area, a disk of the area has the largest 0-torsional rigidity. Such an isoperimetric result can be naturally extended to the γ-torsional rigidity.

Proposition 1. *Given $\gamma\in[0,1)$. Let $(\mathbb{M}^2,\mathrm{g})$ be a complete Riemannian two-manifold with $\tau_{\mathrm{g}}>0$. If u is the solution of (1) with $O\in\mathcal{F}(\mathbb{M}^2)$ being simply-connected, then*

$$\int_O u^{1+\gamma}\,dA_{\mathrm{g}}\le\left(\frac{1+\gamma}{2\tau_{\mathrm{g}}}\right)\left(\int_O u^\gamma\,dA_{\mathrm{g}}\right)^2,\tag{3}$$

equivalently,

$$\int_O\left|\nabla_{\mathrm{g}}u\right|^2_{\mathrm{g}}\,dA_{\mathrm{g}}\le\left(\frac{1+\gamma}{2\tau_{\mathrm{g}}}\right)\left(\int_{\partial O}\left|\nabla_{\mathrm{g}}u\right|_{\mathrm{g}}\,dL_{\mathrm{g}}\right)^2.\tag{4}$$

Moreover, if $\mathbb{M}^2=\mathbb{R}^2$ and $O=B_{\mathrm{g}}(o,r)$, then equality of (3) or (4) is valid.

Proof. Partially inspired by R. Sperb's exposition in [18, pp. 190-196], we make the following argument.

Given a simply-connected domain $O \in \mathcal{F}(\mathbb{M}^2)$. For $0 \le t \le S := \sup_{z\in O} u(z)$ let

$$O_t = \{z \in O : u(z) > t\};\ \ \partial O_t = \{z \in O : u(z) = t\};\ \ a(t) = A_{\mathrm{g}}(O_t).$$

Without loss of generality, we may assume that the set of the critical points of u is finite. An application of the well-known co-area formula gives

$$\frac{da(t)}{dt} = -\int_{\partial O_t} \big|\nabla_{\mathrm{g}} u\big|_{\mathrm{g}}^{-1}\, dL_{\mathrm{g}}. \tag{5}$$

Using (5), Cauchy-Schwarz's inequality and $\tau_{\mathrm{g}} > 0$, we find

$$\tau_{\mathrm{g}} a(t) \le \big(L_{\mathrm{g}}(\partial O_t)\big)^2 \le \Big(-\frac{da(t)}{dt}\Big)\int_{\partial O_t} \big|\nabla_{\mathrm{g}} u\big|_{\mathrm{g}}\, dL_{\mathrm{g}}. \tag{6}$$

For convenience, set

$$I_\gamma(t) = \int_{O_t} u^\gamma\, dA_{\mathrm{g}} \quad \& \quad I_{1+\gamma}(t) = \int_{O_t} u^{1+\gamma}\, dA_{\mathrm{g}}.$$

Then, using the layer-cake formula, the integration-by-part and (5), we get

$$I_\gamma(t) = \int_t^S \Big(\int_{\partial O_s} \big|\nabla_{\mathrm{g}} u\big|_{\mathrm{g}}^{-1}\, dL_{\mathrm{g}}\Big) s^\gamma\, ds,$$

whence finding

$$\frac{dI_\gamma(t)}{dt} = -t^\gamma \int_{\partial O_t} \big|\nabla_{\mathrm{g}} u\big|_{\mathrm{g}}^{-1}\, dL_{\mathrm{g}} = t^\gamma\Big(\frac{da(t)}{dt}\Big)$$

and so

$$\frac{dI_\gamma(t)}{da(t)} = t^\gamma. \tag{7}$$

On the other hand, an application of (6), Green's formula, (1), and $\tau_{\mathrm{g}} > 0$ implies

$$I_\gamma(t) = -\int_{O_t} \Delta_{\mathrm{g}} u\, dA_{\mathrm{g}} = \int_{\partial O_t} \big|\nabla_{\mathrm{g}} u\big|_{\mathrm{g}}\, dL_{\mathrm{g}} \ge \tau_{\mathrm{g}} a(t)\Big(-\frac{dt}{da(t)}\Big). \tag{8}$$

By (7)-(8) we obtain

$$I_\gamma(t)\Big(\frac{dI_\gamma(t)}{da(t)}\Big) + \tau_{\mathrm{g}} t^\gamma a(t)\Big(\frac{dt}{da(t)}\Big) \ge 0. \tag{9}$$

Now, choosing $a = a(t)$ as an independent variable, we get $A = a(0)$ and $0 = a(S)$. Then, integrating (9) over the interval $(0, A)$, taking an

integration-by-part, and using (5) once again, as well as the layer-cake formula, we achieve

$$\begin{aligned} 0 &\leq \int_0^A \Big(\frac{dI_\gamma}{da}\Big) I_\gamma\, da + \tau_{\mathsf{g}} \int_0^A a t^\gamma \Big(\frac{dt}{da}\Big)\, da \\ &= 2^{-1}\int_0^A dI_\gamma^2 - \Big(\frac{\tau_{\mathsf{g}}}{1+\gamma}\Big)\int_0^A t^{1+\gamma}\, da \\ &= 2^{-1}\big(I_\gamma(0)\big)^2 - \Big(\frac{\tau_{\mathsf{g}}}{1+\gamma}\Big)\int_0^S t^{1+\gamma}\Big(\int_{\partial O_t} \big|\nabla_{\mathsf{g}} u\big|_{\mathsf{g}}^{-1}\, dL_{\mathsf{g}}\Big)\, dt \\ &= 2^{-1}\big(I_\gamma(0)\big)^2 - \Big(\frac{\tau_{\mathsf{g}}}{1+\gamma}\Big) I_{1+\gamma}(0), \end{aligned}$$

thereby finding (3) right away.

Clearly, (4) follows from (3) and

$$\int_O u^\gamma\, dA_{\mathsf{g}} = -\int_O \Delta_{\mathsf{g}} u\, dA_{\mathsf{g}} = -\int_{\partial O} \frac{\partial u}{\partial \nu}\, dL_{\mathsf{g}} = \int_{\partial O} |\nabla_{\mathsf{g}} u|\, dL_{\mathsf{g}}$$

in which the Green formula has been used and $\partial/\partial\nu$ represents the partial derivative along the unit outward normal to the boundary ∂O.

The equality case of (3) or (4) under $\mathbb{M}^2 = \mathbb{R}^2$ and $O = B_{\mathsf{e}}(o,r)$ (the origin-centered disk of radius r) can be verified via a direct calculation with the radial solution u (cf. [7]) to

$$\begin{cases} \Delta_{\mathsf{e}} u = -\kappa_\gamma u^\gamma \ \ \&\ \ u > 0 \ \text{ in } \ B_{\mathsf{e}}(o,r); \\ u|_{\partial B_{\mathsf{e}}(o,r)} = 0 \ \text{ and } \ \int_{B_{\mathsf{e}}(o,r)} u^{1+\gamma}\, dA_{\mathsf{e}} = 1, \end{cases}$$

where

$$\kappa_\gamma := \inf_{v \in W_0^{1,2}(B_{\mathsf{e}}(o,r))} \Big\{ \int_{B_{\mathsf{e}}(o,r)} \big|\nabla_{\mathsf{e}} v\big|_{\mathsf{e}}^2\, dA_{\mathsf{e}} : \quad \int_{B_{\mathsf{e}}(o,r)} |v|_{\mathsf{e}}^{1+\gamma}\, dA_{\mathsf{e}} = 1 \Big\}.$$

□

Remark 2. *Under the same hypothesis on* $(\mathbb{M}^2, \mathsf{g})$ *as Proposition 1, we can discover two interesting facts:*
(i) *If* $\gamma = 0$, $K_{\mathsf{g}} \geq 0$, *and*

$$\inf_{(o,r)\in\mathbb{M}^2\times(0,\infty)} \frac{2\tau_{\mathsf{g}}\mathcal{T}_{0,\mathsf{g}}\big(B_{\mathsf{g}}(o,r)\big)}{(\pi r^2)^2} \geq 1$$

which, plus the special case $\gamma = 0$ *of (3), implies*

$$\inf_{(o,r)\in\mathbb{M}^2\times(0,\infty)} \frac{A_{\mathsf{g}}\big(B_{\mathsf{g}}(o,r)\big)}{\pi r^2} \geq 1,$$

then $\mathbb{M}^2$ *is isometric to* $\mathbb{R}^2$ *due to E. Hebey's explanation on* [12, p. 244].

(ii) *When $\gamma = 1$, the corresponding formulation of (3) (cf.* [18, p. 195, (11.24)] *for $\mathbb{M}^2 = \mathbb{R}^2$) is: if u denotes the Laplace-Beltrami eigenfunction associated to $\Lambda_{\mathrm{g}}(O)$, then*

$$\int_O u^2\,dA_{\mathrm{g}} \le \frac{\Lambda_{\mathrm{g}}(O)}{\tau_{\mathrm{g}}}\left(\int_O u\,dA_{\mathrm{g}}\right)^2, \tag{10}$$

amounting to,

$$\int_O \big|\nabla u\big|_{\mathrm{g}}^2\,dA_{\mathrm{g}} \le \frac{1}{\tau_{\mathrm{g}}}\left(\int_{\partial O}\big|\nabla_{\mathrm{g}} u\big|_{\mathrm{g}}\,dL_{\mathrm{g}}\right)^2. \tag{11}$$

Moreover, equality in (10) or (11) holds for $\mathbb{M}^2 = \mathbb{R}^2$ and $O = B_{\mathsf{e}}(o,r)$.

3. VARIATION

Following the first variation formula of the principal eigenvalue (i.e., $\gamma = 1$) discovered in P. R. Garabedian and M. Schiffer [10] when $\mathbb{M}^2 = \mathbb{R}^2$ and in A. El Soufi and S. Ilias [9] for the general setting which was reformulated by F. Pacard and P. Sicbaldi in [14, Proposition 2.1], we can establish an extension from Λ_{g} to $\mathcal{T}_{\gamma,\mathrm{g}}$ with $\gamma \in [0,1)$.

Proposition 3. *Let $\gamma \in [0,1)$ and $(\mathbb{M}^2, \mathrm{g})$ be a complete Riemannian two-manifold. For a given time interval $|t| < t_0$ suppose that $O_t = \xi(t, O_0)$ is the flow on a domain $O_0 \in \mathcal{F}(\mathbb{M}^2)$ associated to the vector field $\Xi(t,z)$, i.e,*

$$\begin{cases} \partial_t(t,z) = \Xi\big(\xi(t,z)\big); \\ \quad \xi(0,z) = z \in O_0. \end{cases} \tag{12}$$

If u_t is the solution of (1) with O replaced by O_t and ν_t is the unit outward normal vector field to ∂O_t, then

$$\frac{d}{dt}\mathcal{T}_{\gamma,\mathrm{g}}(O_t)\Big|_{t=0} = \left(\frac{1+\gamma}{1-\gamma}\right)\int_{\partial O_0}\langle\nabla_{\mathrm{g}} u_0, \nu_0\rangle_{\mathrm{g}}^2\langle\nabla_{\mathrm{g}}\Xi, \nu_0\rangle_{\mathrm{g}}\,dL_{\mathrm{g}}. \tag{13}$$

Proof. Note that $u_t\big(\xi(t,z)\big) = 0$ holds for any $z \in \partial O_0$. So, a differentiation with respect to $t = 0$ gives $\partial_t u_0\big|_{t=0} = -\langle\nabla_{\mathrm{g}} u_0, \Xi\rangle_{\mathrm{g}}$ on ∂O_0. Because u_0 vanishes on ∂O_0, only the normal component of Ξ plays a role in the last formula. As a result, one gets

$$\partial_t u_0\big|_{t=0} = -\langle\nabla_{\mathrm{g}} u_0, \nu_0\rangle_{\mathrm{g}} = \langle\Xi, \nu_0\rangle_{\mathrm{g}} \quad \text{on} \quad \partial O_0. \tag{14}$$

Next, since $-\Delta_{\mathrm{g}} u_t = u_t^\gamma$ holds in O_t, taking the partial derivative of this last equation at $t = 0$ yields

$$0 = \Delta_{\mathrm{g}}\partial_t u_0\big|_{t=0} + \gamma u_0^{\gamma-1}\partial_t u_0\big|_{t=0} \quad \text{in} \quad O_0. \tag{15}$$

Now, an application of the definition of $\mathcal{T}_{\gamma,\mathsf{g}}(O_t)$, (14), (15), (1) with O_0, and Green's formula derives

$$\begin{aligned} \frac{d}{dt}\mathcal{T}_{\gamma,\mathsf{g}}(O_t)\Big|_{t=0} &= (\gamma+1)\int_{O_0} u^\gamma \partial_t u_0\big|_{t=0}\, dA_{\mathsf{g}} \\ &= \Big(\frac{\gamma+1}{\gamma-1}\Big)\int_{O_0}\Big(\partial_t u_0\big|_{t=0}\Delta_{\mathsf{g}} u_0 - u_0\Delta_{\mathsf{g}}\partial_t u_0\big|_{t=0}\Big)\, dA_{\mathsf{g}} \\ &= \Big(\frac{1+\gamma}{1-\gamma}\Big)\int_{\partial O_0}\langle\nabla_{\mathsf{g}} u_0,\nu_0\rangle_{\mathsf{g}}^2\langle\nabla_{\mathsf{g}}\Xi,\nu_0\rangle_{\mathsf{g}}\, dL_{\mathsf{g}}. \end{aligned}$$

Finally, (13) follows. □

Remark 4. *Two comments are in order:*
(i) *Under* $\mathbb{M}^2=\mathbb{R}^2$ *and* $\gamma=0$, *an early form of (13) was established by J. Hadamard* [11] *(cf.* [13]*), but also a convex-body-based variant of (13) was stated in A. Colesanti* [6, Proposition 18].
(ii) *Clearly, (13) does not allow* $\gamma=1$ *whose corresponding formula for the principal eigenvalue is the following: (cf.* [14, Proposition 2.1]*):*

$$\frac{d}{dt}\Lambda_{\mathsf{g}}(O_t)\Big|_{t=0} = -\int_{\partial O_0}\langle\nabla_{\mathsf{g}} u_0,\nu_0\rangle_{\mathsf{g}}^2\langle\nabla_{\mathsf{g}}\Xi,\nu_0\rangle_{\mathsf{g}}\, dL_{\mathsf{g}}. \tag{16}$$

Of course, O_t *in (16) is generated by the solution* u_t *of (2) with* λ *replaced by* $\Lambda_{\mathsf{g}}(O_t)$.

4. Monotonicity

According to [6, p. 132], we have that if $\mathbb{M}^2=\mathbb{R}^2$, $\mathsf{g}=\mathsf{e}$, and O is a convex domain containing the origin in its interior, then $v_r(z)=r^{\frac{2}{1-\gamma}}u(r^{-1}z)$ solves (1) with O replaced by its r-dilation rO and hence

$$\mathcal{T}_{\gamma,\mathsf{e}}(rO)=\int_{rO}|\nabla_{\mathsf{e}} v|_{\mathsf{e}}^2\, dA_{\mathsf{e}} = r^{\frac{4}{1-\gamma}}\int_O |\nabla_{\mathsf{e}} u|_{\mathsf{e}}^2\, dA_{\mathsf{e}} = r^{\frac{4}{1-\gamma}}\mathcal{T}_{\gamma,\mathsf{e}}(O). \tag{17}$$

This observation leads to the following monotonicity formula for the γ-torsional rigidity of a geodesic disk.

Proposition 5. *Given* $\gamma\in[0,1)$. *Let* $(\mathbb{M}^2,\mathsf{g})$ *be a complete Riemannian two-manifold with* $K_{\mathsf{g}}\ge 0$ *and* $\tau_{\mathsf{g}}>0$. *If* $o\in\mathbb{M}^2$ *is fixed, then*

$$r\mapsto \mathcal{Q}_{\gamma,\mathsf{g}}(o,r):=\frac{\mathcal{T}_{\gamma,\mathsf{g}}(B_{\mathsf{g}}(o,r))}{r^{\frac{\tau_{\mathsf{g}}}{\pi(1-\gamma)}}}$$

is monotone increasing in $(0,\infty)$. *Consequently,*

$$\lim_{r\downarrow 0}\mathcal{Q}_{\gamma,\mathsf{g}}(o,r)\le \mathcal{Q}_{\gamma,\mathsf{g}}(o,r)\le \lim_{r\uparrow\infty}\mathcal{Q}_{\gamma,\mathsf{g}}(o,r)\quad \forall\quad r\in(0,\infty)$$

holds with equalities for $\mathbb{M}^2=\mathbb{R}^2$.

Proof. Suppose that u is the solution of (1) with $O = B_{\rm g}(o,r)$. Since $K_{\rm g} \ge 0$, a generalized version of the well-known Bishop-Gromov comparison theorem (cf. [15, p. 41, Theorem 2.14]) yields

$$\frac{d}{dr}\Big(r^{-1}L_{\rm g}\big(\partial B_{\rm g}(o,r)\big)\Big) \le 0 \quad \& \quad L_{\rm g}\big(\partial B_{\rm g}(o,r)\big) \le 2\pi r. \tag{18}$$

Applying $\tau_{\rm g} > 0$, (4), Green's formula, Cauchy-Schwarz's inequality, and (18), we get

$$\begin{aligned} \mathcal{T}_{\gamma,{\rm g}}\big(B_{\rm g}(o,r)\big) &\le \Big(\frac{1+\gamma}{2\tau_{\rm g}}\Big)\Big(\int_{\partial B_{\rm g}(o,r)} |\nabla_{\rm g} u|_{\rm g}\, dL_{\rm g}\Big)^2 \\ &\le \Big(\frac{1+\gamma}{2\tau_{\rm g}}\Big) L_{\rm g}\big(\partial B_{\rm g}(o,r)\big) \int_{\partial B_{\rm g}(o,r)} |\nabla_{\rm g} u|_{\rm g}^2\, dL_{\rm g} \\ &\le \Big(\frac{1+\gamma}{(\pi r)^{-1}\tau_{\rm g}}\Big) \int_{\partial B_{\rm g}(o,r)} |\nabla_{\rm g} u|_{\rm g}^2\, dL_{\rm g} \end{aligned} \tag{19}$$

On the other hand, consider a vector field induced by a normal shift $\delta\nu$, counted positively in the direction of the outward normal to $\partial B_{\rm g}(o,r)$. More precisely, for $t > -r$ and $z \in \partial B_{\rm g}(o,r)$ let $\xi = \xi(t,z)$ be the point on the geodesic radius starting at o of $B_{\rm g}(o,r)$ with $d_{\rm g}(o,\xi) = (1 + tr^{-1})d_{\rm g}(o,z)$. Consequently, if $B_{\rm g}(o,r)$ is chosen as the initial domain O_0 in Proposition 3, then

$$\xi(0, B_{\rm g}(o,r)) = O_0 \quad \& \quad \xi(t, B_{\rm g}(o,r)) = O_t = B_{\rm g}(o, r+t).$$

Once setting $\Xi(\xi(t,z))$ be the point on the geodesic (radial) direction from o to $\xi(t,z)$ with $(r+t)^{-1}d_{\rm g}(o,\xi)$ as its distance from o, we see that (12) holds. Obviously, the unit outward normal vector to the boundary ∂O_0 at $\xi \in \partial O_0$ is the unit vector formed by ξ and so equal to $\Xi(\xi)$. Suppose now that u is the solution of (1) with $O = B_{\rm g}(o,r)$. Then, an application of (13) gives

$$\frac{d}{dr}\mathcal{T}_{\gamma,{\rm g}}\big(B_{\rm g}(o,r)\big) = \Big(\frac{1+\gamma}{1-\gamma}\Big)\int_{\partial B_{\rm g}(o,r)} \big|\nabla_{\rm g} u\big|_{\rm g}\, dL_{\rm g}. \tag{20}$$

Next, we employ (19) and (20) to achieve

$$\frac{d}{dr}\mathcal{Q}_{\gamma,{\rm g}}(r) = \frac{r\frac{d}{dr}\mathcal{T}_{\gamma,{\rm g}}\big(B_{\rm g}(o,r)\big) - \big(\frac{\tau_{\rm g}}{\pi(1-\gamma)}\big)\mathcal{T}_{\gamma,{\rm g}}\big(B_{\rm g}(o,r)\big)}{r^{1-\frac{\tau_{\rm g}}{\pi(1-\gamma)}}} \ge 0,$$

thereby reaching the desired monotonicity. Of course, the consequence part is immediate. □

Remark 6. *When $\gamma = 1$, by (16) and the foregoing proof we can establish that under the same hypothesis on $(\mathbb{M}^2, \mathrm{g})$ as in Proposition 5,*

$$r \mapsto \mathcal{Q}_{\mathsf{g}}(o,r) := \frac{\Lambda_{\mathsf{g}}\big(B_{\mathsf{g}}(o,r)\big)}{r^{-\frac{\tau_{\mathsf{g}}}{2\pi}}}$$

is monotone decreasing in $(0,\infty)$. Consequently,

$$\lim_{r\uparrow\infty} \mathcal{Q}_{\mathsf{g}}(o,r) \le \mathcal{Q}_{\mathsf{g}}(o,r) \le \lim_{r\downarrow 0} \mathcal{Q}_{\mathsf{g}}(o,r) \quad \forall \quad r \in (0,\infty)$$

holds with equalities for $\mathbb{M}^2 = \mathbb{R}^2$ – this follows immediately from the well-known fact (see e.g. [6, p. 110]*) that Λ_{e} is homogeneous of order -2.*

5. Appendix

In their 2008 paper [3], R. Burckel, D. Marshall, D. Minda, P. Poggi-Corradini and T. Ransford discovered the area-capacity-diameter versions of the following Schwarz's lemma variant: For a holomorphic map f from the origin-centered unit disk $B_{\mathsf{e}}(o,1)$ into $\mathbb{R}^2$,

$$r \mapsto \frac{\sup_{z\in B_{\mathsf{e}}(o,r)} |f(z) - f(o)|_{\mathsf{e}}}{r}$$

is strictly increasing in $(0,1)$ unless f is linear. Soon after, their results were extended differently by A. Y. Solynin [17], D. Betsakos [1]-[2], and J. Xiao and K. Zhu [19]. While, as a new complement to [3], T. Carroll and J. Ratzkin's 2010 article [4] on the Schwarz type lemma for Λ_{e} has partially stimulated us to carry out our current project. In contrast to the monotone-decreasing-principle (i.e., the backward Schwarz type lemma) in [4] saying that

$$r \mapsto \frac{\Lambda_{\mathsf{e}}\big(f(B_{\mathsf{e}}(o,r))\big)}{\Lambda_{\mathsf{e}}(B_{\mathsf{e}}(o,r))}$$

is strictly decreasing in $(0,1)$ unless f is a linear map, we have the forward Schwarz type lemma for the γ-torsional rigidity:

Lemma 7. *Given $\gamma \in [0,1)$. If f is a conformal mapping from $B_{\mathsf{e}}(o,1)$ into $\mathbb{R}^2$, then*

$$r \mapsto \Phi_{\gamma,\mathsf{e}}(f;r) := \frac{\mathcal{T}_{\gamma,\mathsf{e}}\big(f(B_{\mathsf{e}}(o,r))\big)}{\mathcal{T}_{\gamma,\mathsf{e}}(B_{\mathsf{e}}(o,r)}$$

is strictly increasing in $(0,1)$ unless f is linear. Consequently,

$$\lim_{r\downarrow 0} \Phi_{\gamma,\mathsf{e}}(f;r) \le \Phi_{\gamma,\mathsf{e}}(f;r) \le \lim_{r\uparrow 1} \Phi_{\gamma,\mathsf{e}}(f;r) \quad \forall \quad r \in (0,1)$$

holds with equalities when f is linear.

Proof. The argument for the monotonicity of $\mathcal{Q}_{\gamma,\mathsf{e}}(f;r)$ in $(0,1)$ is similar to that proving [4, Theorem 1]. The key point is to construct a proper vector field via the given conformal map f. More precisely, if g stands for the inverse map of f, then

$$\xi = \xi(t,w) = f\big((1+tr^{-1})g(w)\big) \quad \forall \quad w \in f(B_{\mathsf{e}}(o,r))$$

and

$$\Xi(\xi) = \frac{g(\xi)f'\big(g(\xi)\big)}{r+t}$$

are selected for (12). Note that the unit outward normal vector to the boundary $\partial f(B_{\mathsf{e}}(o,r))$ at ξ is

$$\nu(\xi) = \Big(\frac{g(\xi)}{r}\Big)\left(\frac{f'\big(g(\xi)\big)}{|f'\big(g(\xi)\big)|_{\mathsf{e}}}\right)$$

and so that

$$\langle \Xi, \nu\rangle_{\mathsf{e}} = |f'(g(\xi))|_{\mathsf{e}} \quad \forall \quad \xi \in \partial f(B_{\mathsf{e}}(o,r)).$$

Next, suppose that u_r is the solution of (1) with $O = f(B_{\mathsf{e}}(o,r))$. Then the chain rule yields

$$|\nabla_{\mathsf{e}} u_r(\xi)|_{\mathsf{e}} = \big|\nabla_{\mathsf{e}} u_r\big(f(z)\big)\big|_{\mathsf{e}} |f'(z)|_{\mathsf{e}} \quad \forall \quad \xi = f(z) \in f(B_{\mathsf{e}}(o,r)),$$

whence giving (by Proposition 3):

$$\frac{d}{dr}\mathcal{T}_{\gamma,\mathsf{e}}\big(f(B_{\mathsf{e}}(o,r))\big) = \Big(\frac{1+\gamma}{1-\gamma}\Big)\int_{\partial B_{\mathsf{e}}(o,r)} |\nabla_{\mathsf{e}} u_r|^2_{\mathsf{e}}\, dL_{\mathsf{e}}. \tag{21}$$

Meanwhile, Proposition 1 plus Cauchy-Schwarz's inequality derives

$$\mathcal{T}_{\gamma,\mathsf{e}}\big(f(B_{\mathsf{e}}(o,r))\big) \le \Big(\frac{1+\gamma}{4r^{-1}}\Big)\int_{\partial B_{\mathsf{e}}(o,r)} |\nabla_{\mathsf{e}} u_r|^2_{\mathsf{e}}\, dL_{\mathsf{e}}. \tag{22}$$

Finally, putting (17), (21) and (22) together, we get that $\frac{d}{dr}\mathcal{Q}_{\gamma,\mathsf{e}}(f;r) \ge 0$ holds with the strict inequality unless f is linear, whence reaching the desired result. Since the consequence part is straightforward, our proof is complete.

□

Remark 8. *Lemma 7 can be extended to a slightly general form: For a holomorphic map f from $B_{\mathsf{e}}(o,1)$ into $\mathbb{R}^2$, let $\mathbf{f}(B_{\mathsf{e}}(o,r))$ be its Riemann surface with constant Gauss curvature -1. Then*

$$r \mapsto \frac{\mathcal{T}_{\gamma,\mathsf{e}}\big(\mathbf{f}(B_{\mathsf{e}}(o,r))\big)}{\mathcal{T}_{\gamma,\mathsf{e}}(B_{\mathsf{e}}(o,r))}$$

is strictly increasing in $(0,1)$ *unless* f *is linear. This is in contrast to* [4, Corollary 2] *which reads as:*

$$r \mapsto \frac{\Lambda_{\mathsf{e}}\big(\mathbf{f}(B_{\mathsf{e}}(o,r))\big)}{\Lambda_{\mathsf{e}}(B_{\mathsf{e}}(o,r))}$$

is strictly decreasing in $(0,1)$ *unless* f *is linear.*

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN'S, NL A1C 5S7, CANADA
E-mail address: jxiao@mun.ca